

BANACH ALGEBRAS, SAMELSON PRODUCTS, AND THE WANG DIFFERENTIAL

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ABSTRACT. Suppose given a principal G bundle $\zeta : P \rightarrow S^k$ (with $k \geq 2$) and a Banach algebra B on which G acts continuously. Let $A_\zeta = \Gamma(P \times_G B \rightarrow S^k)$ denote the associated Banach algebra of sections. Then $\pi_* \mathrm{GL} A_\zeta$ is determined by a mostly degenerate spectral sequence and by a Wang differential

$$d_k : \pi_* (\mathrm{GL} B) \longrightarrow \pi_{*+k} (\mathrm{GL} B).$$

We show that the differential is given explicitly in terms of an enhanced Samelson product with the clutching map of the principal bundle. Analogous results hold after localization and in the setting of topological K -theory. We illustrate our technique with a close analysis of the C^* -algebra of sections of the bundle

$$S^7 \times_{S^3} M_2(\mathbb{C}) \rightarrow S^4$$

constructed from the Hopf bundle $S^7 \rightarrow S^4$ and by the conjugation action of S^3 on $M_2(\mathbb{C})$.

CONTENTS

1. Introduction	1
2. The spectral sequences	3
3. The spectral sequences when X is a sphere	5
4. Deriving the Wang sequence	6
5. Identifying the differential	7
6. Restrictions give fibrations	8
7. Enhanced Samelson products	10
8. Identifying the differential more precisely	11
9. Proofs of the Main Theorems	13
10. An Example	15
References	18

1. INTRODUCTION

The group $\mathrm{GL} B$ of invertible elements of a Banach algebra has the homotopy type of a CW-complex and hence its homotopy groups are in principle computable. We know that these groups hold a lot of information about B , since the topological K -theory groups $K_*(B)$ are given by the stabilized groups $\pi_*(\mathrm{GL}(B \otimes \mathcal{K}))$. The groups $\pi_*(\mathrm{GL} B)$ are far richer in information but also far more difficult to compute.

2010 *Mathematics Subject Classification.* 46L80, 46L85, 46M20, 55Q52, 55R20, 55T25 .

Key words and phrases. general linear group of a Banach algebra, Atiyah-Hirzebruch spectral sequence, spectral sequence differential, K -theory for Banach algebras, unstable K -theory, Samelson product, enhanced Samelson product, Wang sequence, Wang differential.

In joint work with Emmanuel Dror-Farjoun, we have developed a spectral sequence aimed at computing these groups in a wide variety of settings. Spectral sequences reveal and conceal. On the one hand, there are long lists of spectacular results in topology and algebra that have been obtained by spectral sequence techniques (cf. [14]). On the other hand, spectral sequence cognoscenti will testify that there is depressingly little known in general about differentials in spectral sequences, so that reducing a problem to a “spectral sequence calculation” may not in fact solve the problem at all. In order to put the results of this paper in context, we review briefly what is known about differentials in this genre of spectral sequence.

The classical Atiyah - Hirzebruch spectral sequence [2] takes the form

$$E_2 = H^*(X; K^*(pt)) \implies K^*(X).$$

Atiyah and Hirzebruch noted the following:

- (1) $d_{2n} = 0$ for all n because $K^{odd}(pt) = 0$.
- (2) d_3 is associated with the integral Bockstein operation

$$Sq^3 : H^k(X; \mathbb{Z}) \rightarrow H^{k+3}(X; \mathbb{Z}).$$

- (3) For $k \geq 2$, each d_k takes values in the torsion subgroup of E_k and hence the spectral sequence collapses rationally:

$$E_2 \otimes \mathbb{Q} \cong E_\infty \otimes \mathbb{Q}.$$

Arlettaz [1] gives explicit integers governing the order of the torsion subgroups. In addition, there are always “dimension” arguments in particular cases. For example, if $H^*(X; \mathbb{Z}) = 0$ for all $*$ odd, then $E_2 = 0$ in odd total degree, $d_j = 0$ for all $j \geq 2$, and so $E_2 \cong E_\infty$.

More generally, if h^* is a generalized cohomology theory then there is a well-known spectral sequence

$$E_2 = H^*(X; h^*(pt)) \implies h^*(X).$$

In another direction there is the classical Federer spectral sequence [8]

$$E_2 = H^*(X; \pi_*(Y)) \implies H^*(F(X, Y)),$$

where $F(X, Y)$ denotes the function space of maps from X to Y with the compact-open topology. Sam Smith [21] shows that in the context of Quillen minimal models, differentials are related to Whitehead products. (Our results will echo this result in the integral situation.)

Moving to twisted K -theory $K_\Delta^*(X)$ associated to a principal bundle, the bundle is classified by its Dixmier-Douady invariant Δ and the spectral sequence takes the form

$$E_2 = H^*(X; K^*(pt)) \implies K_\Delta^*(X)$$

with differential related to the Dixmier-Douady invariant by the result of J. Rosenberg [18]. Atiyah and Segal ([3] Prop. 7.5) show that if the base space is a compact manifold then in the associated spectral sequence

$$E_2 = H^*(X; K^*(pt)) \otimes \mathbb{R} \implies K_\Delta^*(X) \otimes \mathbb{R},$$

all higher ($j \geq 4$) differentials are given by Massey products. They point out that this implies that all higher differentials vanish (over the reals) when the base space is a compact Kähler manifold, by the deep result of [6].

We consider the case of a fibre bundle over a sphere in order to isolate a new type of differential. Here are our primary results.

Theorem A. *Suppose that $X = S^k$ with $k \geq 2$ and that $\zeta : E = P \times_G B \rightarrow S^k$ is a bundle of Banach algebras. Let A_ζ denote the associated section algebra. Then there is a long exact Wang¹ sequence*

$$\cdots \rightarrow \pi_n(\mathrm{GL}_0 A) \otimes \mathbb{P} \xrightarrow{r} \pi_n(\mathrm{GL}_0 B) \otimes \mathbb{P} \xrightarrow{d_k} \pi_{n+k-1}(\mathrm{GL}_0 B) \otimes \mathbb{P} \xrightarrow{s} \pi_{n-1}(\mathrm{GL}_0 A) \otimes \mathbb{P} \rightarrow \cdots$$

The differential d_k is given by

$$d_k(1 \otimes a) = -g \otimes [\kappa, a]$$

where g is the generator of $H^k(S^k; \mathbb{Z})$ and $[\kappa, a]$ is the enhanced Samelson product² with the class $\kappa : S^{k-1} \rightarrow G$ that classifies the principal bundle.

Passing to limits, we also obtain the analogous sequence at the level of K -theory:

Theorem B. *Suppose that $X = S^k$ with $k \geq 2$ and that $\zeta : E = P \times_G B \rightarrow S^k$ is a bundle of Banach algebras. Let A_ζ denote the associated section algebra. Then there is a long exact sequence*

$$\cdots \rightarrow K_n(A) \otimes \mathbb{P} \xrightarrow{\rho} K_n(B) \otimes \mathbb{P} \xrightarrow{d_k} K_{n+k-1}(B) \otimes \mathbb{P} \xrightarrow{\sigma} K_{n-1}(A) \otimes \mathbb{P} \rightarrow \cdots$$

When $k = 3$ then d_3 is given by multiplication by $-\Delta_\zeta \beta$ where Δ_ζ is the Dixmier-Douady integer and β is Bott periodicity.

The enhanced Samelson product referred to is a generalization of the classical Samelson product that we explain in §7.

The paper is organized as follows. First we state the general spectral sequence results of [7], we specialize them to the case when the base is a sphere, and we derive the homological algebra version of the Wang sequence. We then make a preliminary identification of the differential. Then we make two detours to classical homotopy theory to verify that certain restriction maps are fibrations and to define the enhanced Samelson product. Next, we put everything together to establish Theorems A and B.

Finally, we illustrate our technique with a close analysis of the C^* -algebra of sections of the bundle

$$S^7 \times_{S^3} M_2(\mathbb{C}) \rightarrow S^4$$

constructed from the Hopf bundle $S^7 \rightarrow S^4$ and by the conjugation action of S^3 on $M_2(\mathbb{C})$. We explicitly compute $\pi_n(U_o A_\zeta)$ for $n \leq 8$ and contrast these results with the computation of $\pi_*(U_o A_\zeta) \otimes \mathbb{Q}$ and with $K_*(A_\zeta)$.

It is a pleasure to thank my collaborators and colleagues Emmanuel Dror-Farjoun, Sam Smith, Dan Isaksen and John Klein for continued assistance and to acknowledge the insight that I received from the work of C. Wockel.

2. THE SPECTRAL SEQUENCES

We recall for reference the main results of Farjoun-Schochet [7].

Suppose that X is a finite dimensional compact metric space and $\zeta : P \rightarrow X$ is a standard³ principal G bundle for some topological group G that acts continuously

¹ Wang's original 1949 paper [26] gave a direct and elementary proof of a homology version of this sequence. Serre ([20] p. 471) put the result into the spectral sequence setting. Since then it has appeared in many contexts.

²The enhanced Samelson product is a generalization of the Samelson product that we define and illustrate in §7.

³A principal G -bundle is *standard* if X is a finite complex or if the bundle is a pullback of a principal G -bundle over some CW -complex. See [7] for details.

on a Banach algebra B via $\alpha : G \rightarrow \text{Aut}(B)$. Let $E = P \times_G B \rightarrow X$ be the associated fibre bundle. (We refer to this set-up as a *standard bundle of Banach algebras*.) Let

$$A_\zeta = \Gamma(X, E)$$

denote the set of continuous sections of the bundle with pointwise operations. This has a natural structure of a Banach algebra. If B is unital, then A_ζ is also unital, with identity the canonical section that to each point $x \in X$ assigns the identity in E_x .

We are interested in $\text{GL } A_\zeta$, the group of invertible elements in A_ζ . (If A_ζ is not unital, then we understand this to mean the kernel of the natural map $\text{GL}(A_\zeta^+) \rightarrow \text{GL}(\mathbb{C})$.) This is a space of the homotopy type of a CW-complex, second countable if B is separable. It may have many (homeomorphic) path components; let $\text{GL}_o A_\zeta$ denote the path component of the identity.⁴

Let \mathbb{P} denote a subring of the rational numbers. (We allow the cases $\mathbb{P} = \mathbb{Z}$ and $\mathbb{P} = \mathbb{Q}$ as well as intermediate rings.)

Theorem 2.1. *Suppose that X is a finite dimensional compact metric space and that $\zeta : E = P \times_G B \rightarrow X$ is a standard bundle of Banach algebras. Let A_ζ denote the associated algebra of sections. Then:*

- (1) *There is a second quadrant spectral sequence converging to $\pi_*(\text{GL}_o A_\zeta) \otimes \mathbb{P}$ with*

$$E_{-p,q}^2 \cong H^p(X; \pi_q(\text{GL}_o B) \otimes \mathbb{P})$$

and

$$d^r : E_{-p,q}^r \longrightarrow E_{-p-r,q+r-1}^r.$$

- (2) *If X has dimension at most n , then $E^{n+1} = E^\infty$.*
 (3) *The spectral sequence is natural with respect to pullback diagrams*

$$\begin{array}{ccc} f^* P \times_G B & \xrightarrow{f \times 1} & P \times_G B \\ \downarrow f^* \zeta & & \downarrow \zeta \\ X' & \xrightarrow{f} & X \end{array}$$

and associated map $f^ : A_\zeta \longrightarrow A_{f^* \zeta}$.*

- (4) *The spectral sequence is natural with respect to G -equivariant maps*

$$\alpha : B \rightarrow B'$$

of Banach algebras

Generally, this spectral sequence does not collapse, even rationally.

Note that in many cases of interest, for instance $B = M_n(\mathbb{C})$, the groups $\pi_*(\text{GL}_o B)$ are unknown, and so the integral version of the spectral sequence cannot be used directly to compute $\pi_*(\text{GL}_o A_\zeta)$. However, frequently the groups $\pi_*(\text{GL}_o B) \otimes \mathbb{Q}$ are known and hence the rational form of the spectral sequence will be practical.

Using the version of Bott periodicity established by R. Wood [30] and M. Karoubi [11] and taking limits of spectral sequences, we derive the following.

⁴If A is a C^* -algebra then denote the group of unitary elements of A by UA . The inclusion $UA \rightarrow \text{GL } A$ is a homotopy equivalence. So when restricting attention to C^* -algebras one generally uses UA .

Theorem 2.2. *Suppose that X is a finite dimensional compact metric space, B is a Banach algebra, and $\zeta : E = P \times_G B \rightarrow X$ is a standard bundle of Banach algebras. Then there is a second quadrant spectral sequence*

$$E_{-p,q}^2(A_{\zeta_\infty}) \cong H^p(X; K_{q+1}(B) \otimes \mathbb{P}) \implies K_{*+1}(A_\zeta) \otimes \mathbb{P}$$

which is the direct limit of the corresponding spectral sequences converging to

$$\pi_*(\mathrm{GL}_o(A_\zeta \otimes M_n(\mathbb{C})) \otimes \mathbb{P}.$$

If X has dimension at most n then $E^{n+1} = E^\infty$.

This result is due to J. Rosenberg [18] when $B = \mathcal{K}$ and A_ζ is a continuous trace C^* -algebra over a finite complex X .

3. THE SPECTRAL SEQUENCES WHEN X IS A SPHERE

Suppose that $X = S^k$ with $k \geq 2$. Then several things simplify radically. First of all, bundles are automatically standard. Second, X is simply connected and hence the local coefficients in the spectral sequences trivialize. Third, the E^2 term vanishes except in columns $p = 0$ and $p = -k$. This implies that $E^2 = E^k$ and $E^{k+1} = E^\infty$ in the spectral sequences, so that the only possible non-zero higher differential is d_k . Combining these elementary observations we have the following versions of Theorems 2.1 and 2.2.

Theorem 3.1. *Suppose that $\zeta : E = P \times_G B \rightarrow S^k$ is a bundle of Banach algebras with $k \geq 2$. Let A_ζ denote the associated section algebra. Then:*

- (1) *There is a second quadrant spectral sequence converging to $\pi_*(\mathrm{GL}_o A_\zeta) \otimes \mathbb{P}$ with $E^2 = 0$ except for*

$$E_{0,q}^2 = E_{0,q}^k \cong H^0(S^k; \mathbb{Z}) \otimes \pi_q(\mathrm{GL}_o B) \otimes \mathbb{P}$$

$$E_{-k,q}^2 = E_{-k,q}^k \cong H^k(S^k; \mathbb{Z}) \otimes \pi_q(\mathrm{GL}_o B) \otimes \mathbb{P}$$

and the only higher possibly non-zero differential is

$$d^k : E_{0,q}^k \longrightarrow E_{-k,q+k-1}^k.$$

- (2) *Thus $E^{k+1} = E^\infty$.*

Theorem 3.2. *Suppose that $\zeta : E = P \times_G B \rightarrow S^k$ is a bundle of Banach algebras with $k \geq 2$. Then there is a second quadrant spectral sequence converging to $K_{*+1}(A_\zeta) \otimes \mathbb{P}$ with $E^2 = 0$ except for*

$$E_{0,q}^2 = E_{0,q}^k \cong H^0(S^k; \mathbb{Z}) \otimes K_{q+1}(B) \otimes \mathbb{P}$$

$$E_{-k,q}^2 = E_{-k,q}^k \cong H^k(S^k; \mathbb{Z}) \otimes K_{q+1}(B) \otimes \mathbb{P}$$

and the only possibly non-zero higher differential is

$$d^k : E_{0,q}^k \longrightarrow E_{-k,q+k-1}^k.$$

Thus $E^{k+1} = E^\infty$.

4. DERIVING THE WANG SEQUENCE

We may rephrase the conclusion of Theorem 3.1 as asserting the existence of a long exact sequence

$$0 \rightarrow E_{0,q}^\infty \rightarrow E_{0,q}^k \xrightarrow{d^k} E_{-k,q+r-1}^k \rightarrow E_{-k,q+r-1}^\infty \rightarrow 0$$

and after identifications we obtain the exact sequence

$$(*) \quad 0 \rightarrow E_{0,q}^\infty \rightarrow \pi_q(\mathrm{GL}_o B) \otimes \mathbb{P} \xrightarrow{d^k} \pi_{q+k-1}(\mathrm{GL}_o B) \otimes \mathbb{P} \rightarrow E_{-k,q+r-1}^\infty \rightarrow 0.$$

On the other hand, the filtration that creates the spectral sequence in the first place comes from the cell filtration of $X = S^k$ and hence simplifies dramatically to become

$$(**) \quad 0 \longrightarrow E_{-k,n+k}^\infty \longrightarrow \pi_n(\mathrm{GL}_o A_\zeta) \otimes \mathbb{P} \longrightarrow E_{0,n}^\infty \longrightarrow 0$$

with

$$E_{-k,n+k}^\infty \cong H^k(S^k) \otimes \pi_{n+k}(\mathrm{GL}_o B) \otimes \mathbb{P}$$

and

$$E_{0,n}^\infty \cong H^0(S^k) \otimes \pi_n(\mathrm{GL}_o B) \otimes \mathbb{P}.$$

Splice the two sequences $\{*\}$ and $\{**\}$ together as follows. Splicing the sequences at $E_{0,n}^\infty$ gives the composite

$$r : \pi_n(\mathrm{GL}_o A_\zeta) \otimes \mathbb{P} \rightarrow E_{0,n}^\infty \rightarrow E_{0,n}^k \cong \pi_n(\mathrm{GL}_o B) \otimes \mathbb{P}$$

where the map r corresponds to the map that takes an element of A_ζ , regards it as a section, and restricts it to the basepoint. Splicing the sequence at $E_{-k,n+k-1}^\infty$ gives the composite

$$s : \pi_{n+k-1}(\mathrm{GL}_o B) \otimes \mathbb{P} \cong E_{-k,n+k-1}^k \rightarrow E_{-k,n+k-1}^\infty \rightarrow \pi_{n-1}(\mathrm{GL}_o A_\zeta) \otimes \mathbb{P}$$

where the map s corresponds to the inclusion of a pointed section into the space of all sections. (See Proposition 8.1 below.)

We obtain the following generalization of the Wang sequence. [13]

Theorem 4.1. *Suppose that $X = S^k$ with $k \geq 2$ and that $\zeta : E = P \times_G B \rightarrow S^k$ is a bundle of Banach algebras. Let A_ζ denote the associated section algebra. Then there is a long exact sequence*

$$\cdots \rightarrow \pi_n(\mathrm{GL}_o A_\zeta) \otimes \mathbb{P} \xrightarrow{r} \pi_n(\mathrm{GL}_o B) \otimes \mathbb{P} \xrightarrow{d_k} \pi_{n+k-1}(\mathrm{GL}_o B) \otimes \mathbb{P} \xrightarrow{s} \pi_{n-1}(\mathrm{GL}_o A_\zeta) \otimes \mathbb{P} \rightarrow \cdots$$

Passing to limits, we also obtain the analogous sequence at the level of K -theory:

Theorem 4.2. *Suppose that $X = S^k$ with $k \geq 2$ and that $\zeta : E = P \times_G B \rightarrow S^k$ is a bundle of Banach algebras. Let A_ζ denote the associated section algebra. Then there is a long exact sequence*

$$\cdots \rightarrow K_n(A_\zeta) \otimes \mathbb{P} \xrightarrow{r} K_n(B) \otimes \mathbb{P} \xrightarrow{d_k} K_{n+k-1}(B) \otimes \mathbb{P} \xrightarrow{s} K_{n-1}(A_\zeta) \otimes \mathbb{P} \rightarrow \cdots$$

Remark 4.3. This result agrees with the result of Rosenberg [18] on continuous trace algebras over S^3 . He shows there that if d_3 is an isomorphism then $K_*(A_\zeta) = 0$. If $d_3 = 0$ then $K_0(A_\zeta) = \mathbb{Z}$ and $K_1(A_\zeta) = 0$. If d_3 is multiplication by $s \neq 0, \pm 1$ then $K_0(A_\zeta) = 0$ and $K_1(A_\zeta) = \mathbb{Z}/s$.

5. IDENTIFYING THE DIFFERENTIAL

In order to identify the unknown differential in these theorems, we must look at the exact couple that gives rise to the spectral sequence as constructed in [7] §4. Suppose that X is a finite complex. The space of invertible sections of the bundle

$$P \times_G \mathrm{GL}_o B \longrightarrow X$$

is filtered up to homotopy by a descending filtration

$$\dots \mathcal{F}_{p+1} X \hookrightarrow \mathcal{F}_p X \dots$$

(See [7] for details.) The resulting exact couple is given by

$$D_{-p,q}^1 \cong \pi_{q-p}(\mathcal{F}_p X) \otimes \mathbb{P}$$

and

$$E_{-p,q}^1 = \pi_{q-p}((\mathcal{F}_p X)/(\mathcal{F}_{p+1} X)) \otimes \mathbb{P}.$$

The structural maps are given as follows:

- (1) The map $i^1 : D_{p,q}^1 \rightarrow D_{p+1,q-1}^1$ is given by the natural map induced by the filtration:

$$\begin{array}{ccc} D_{-p,q}^1 & \xrightarrow{i^1} & D_{-p+1,q-1}^1 \\ \downarrow \cong & & \downarrow \cong \\ \pi_{q-p}(\mathcal{F}_p X) & \longrightarrow & \pi_{q-p}(\mathcal{F}_{p-1} X) \end{array}$$

- (2) The map $j^1 : D_{-p,q}^1 \rightarrow E_{-p,q}^1$ is given by

$$\begin{array}{ccc} D_{-p,q}^1 & \xrightarrow{j^1} & E_{-p,q}^1 \\ \downarrow \cong & & \downarrow \cong \\ \pi_{q-p}(\mathcal{F}_p X) & \longrightarrow & \pi_{q-p}((\mathcal{F}_p X)/(\mathcal{F}_{p+1} X)) \otimes \mathbb{P} \end{array}$$

- (3) The map $\delta^1 : E_{p,q}^1 \rightarrow D_{p-1,q}^1$ is given by

$$\begin{array}{ccc} E_{-p,q}^1 & \xrightarrow{\delta^1} & D_{-p-1,q}^1 \\ \downarrow \cong & & \downarrow \cong \\ \pi_{q-p}((\mathcal{F}_p X)/(\mathcal{F}_{p+1} X)) \otimes \mathbb{P} & \xrightarrow{\delta} & \pi_{q-p-1}((\mathcal{F}_{p+1} X)) \otimes \mathbb{P} \end{array}$$

with differential

$$d^1 = j^1 \delta^1 : E_{-p,q}^1 \longrightarrow E_{-p-1,q}^1.$$

We may identify the E^1 term by noting that

$$\begin{aligned} E_{-p,q}^1 &\cong \pi_{q-p}(\Gamma(\vee_\alpha S^p, (\mathrm{GL}_o A_\zeta)|_{S^p}) \otimes \mathbb{P} \cong \pi_{q-p}(F_*(\vee_\alpha S^p, \mathrm{GL}_o B)) \otimes \mathbb{P} \cong \\ &\cong \oplus_\alpha \pi_{q-p}(\Omega^p \mathrm{GL}_o B) \otimes \mathbb{P} \cong \oplus_\alpha \pi_q(\mathrm{GL}_o B) \otimes \mathbb{P} \cong C^p(X; \pi_q(\mathrm{GL}_o B) \otimes \mathbb{P}) \end{aligned}$$

so that

$$E_{-p,q}^1 \cong C^p(X; \pi_q(\mathrm{GL}_o B) \otimes \mathbb{P}),$$

the cellular cochains of X with coefficients in $\pi_q(\mathrm{GL}_o B) \otimes \mathbb{P}$. The d^1 differential is the usual cellular differential and so

$$E_{-p,q}^2 \cong \check{H}^p(X; \pi_q(\mathrm{GL}_o B) \otimes \mathbb{P}).$$

However, when $X = S^k$ then the matter becomes a lot simpler- the E^2 term vanishes except for $p = 0, k$. The differentials d^s vanish for $s < k$. Internally, the derived exact couples have the property that the maps

$$D_{-u,v}^s \xrightarrow{i^s} D_{-u+1,v-1}^s \xrightarrow{i^s} \dots \longrightarrow D_{-1,v-u+1}^s$$

are isomorphisms for $s \leq k$. Thus we may identify the d^k differential as the composite

$$E_{0,q}^k \xrightarrow{\delta^k} D_{-1,q}^k \xleftarrow{\cong} \dots \xleftarrow{\cong} D_{-k,q+k-1}^k \xrightarrow{j^k} E_{-k,q+k-1}^k$$

We summarize:

Proposition 5.1. *In the case of Theorem 4.1 where $X = S^k$, the d^k differential is given as the composite*

$$E_{0,q}^k \xrightarrow{\delta^k} D_{-1,q}^k \xleftarrow{\cong} \dots \xleftarrow{\cong} D_{-k,q+k-1}^k \xrightarrow{j^k} E_{-k,q+k-1}^k$$

where

$$\begin{aligned} E_{0,q}^2 &= E_{0,q}^k \cong H^0(S^k; \mathbb{Z}) \otimes \pi_q(\mathrm{GL}_o B) \otimes \mathbb{P} \\ E_{-k,q+k-1}^2 &= E_{-k,q+k-1}^k \cong H^k(S^k; \mathbb{Z}) \otimes \pi_{q+k-1}(\mathrm{GL}_o B) \otimes \mathbb{P}. \end{aligned}$$

□

So what is this map? Its identification requires two topological detours. We must verify that certain maps are fibrations and we must generalize the classical Samelson product. The following two sections deal with these matters.

6. RESTRICTIONS GIVE FIBRATIONS

Suppose that $P \rightarrow X$ is a principal G -bundle. Then any associated bundle $P \times_G B \rightarrow X$ is a locally trivial fibre bundle. If the base space is paracompact then it is a standard fact (cf. [22], Theorem 7.14, page 96) that it is a fibration. Similarly, the fibre bundle $P \times_G \mathrm{GL} B \rightarrow X$ is a fibration.

Theorem 6.1. *Suppose that $E \rightarrow X$ is a standard bundle of Banach algebras over a compact space X . Let X' be a closed subspace and assume that the inclusion $X' \rightarrow X$ is a cofibration. Let $E' \rightarrow X'$ be the induced bundle. Then the restriction map*

$$r : \Gamma(X, E) \longrightarrow \Gamma(X', E')$$

is a fibration.

Proof. Suppose first that $E \rightarrow X$ is a trivial bundle, so that up to fibre homotopy equivalence we may replace it by $X \times B \rightarrow X$ for a fixed Banach algebra B . Then after obvious identification the induced map

$$r : \Gamma(X, E) \longrightarrow \Gamma(X', E')$$

becomes

$$F(X, B) \rightarrow F(X', B)$$

which is a fibration by classical results (cf. [22], Theorem 8.2, page 97.) So the result is true when $E \rightarrow X$ is a trivial fibration.

Now for the general case. Since the bundle $E \rightarrow X$ is locally trivial, we may find finite open covers $\{U_j\}$ and $\{V_j\}$ and closed sets $\{D_j\}$ such that $U_j \subset D_j \subset V_j$

and E is trivial when restricted to each V_j . For each j there is thus a commuting diagram

$$\begin{array}{ccc} \Gamma(X, E) & \xrightarrow{p_j} & \Gamma(D_j, E|_{D_j}) \\ \downarrow r & & \downarrow r_j \\ \Gamma(X', E') & \xrightarrow{p_j} & \Gamma(D_j \cap X', E'|_{D_j \cap X'}) \end{array}$$

and each map r_j is a fibration, by the first part of the proof.

Let W be a space and suppose given a continuous map

$$h_t : W \times [0, 1] \rightarrow \Gamma(X', E')$$

together with a lift of h_0 to

$$H_0 : W \times [0, 1] \rightarrow \Gamma(X, E)$$

with $rH_0 = h_0$. We must lift h_t .

Composing, we have, for each index j ,

$$p_j h_t : W \times [0, 1] \rightarrow \Gamma(D_j \cap X', E'|_{D_j \cap X'})$$

together with a lift of $p_j h_0$ to

$$\tilde{H}_0^j = p_j H_0 : W \times [0, 1] \rightarrow \Gamma(D_j, E|_{D_j})$$

with $r_j H_0 = p_j h_0$. The map r_j is a fibration, for each j , by the first part of the argument, and hence there exist

$$\tilde{H}_t^j = p_j H_t : W \times [0, 1] \rightarrow \Gamma(D_j, E|_{D_j})$$

extending \tilde{H}_0^j and with $r_j \tilde{H}_t^j = p_j h_t$.

Let $\{u_j\}$ be a partition of unity subordinate to $\{U_j\}$. Define

$$H_t : W \times [0, 1] \longrightarrow \Gamma(X, E)$$

by

$$H_t(w)(x) = \sum_j u_j(x) \tilde{H}_t^j(w)(x).$$

This is continuous on its domain. Restricting,

$$\begin{aligned} H_0(w)(x) &= \sum_j u_j(x) \tilde{H}_0^j(w)(x) = \sum_j u_j(x) p_j H_0(w)(x) = \\ &= \sum_{x \in V_j} u_j(x) H_0(w)(x) = H_0(w)(x) \end{aligned}$$

and

$$\begin{aligned} rH_t(w)(x) &= \sum_j u_j(x) r_j \tilde{H}_t^j(w)(x) = \sum_j u_j(x) p_j h_t(w)(x) = \\ &= \sum_j u_j(x) h_t(w)(x) = h_t(w)(x) \end{aligned}$$

as required. \square

Corollary 6.2. *Suppose that $E \rightarrow X$ is a standard bundle of Banach algebras over a compact space X . Let $x_0 \in X$ be a non-degenerate basepoint.⁵ Then the induced evaluation map*

$$r : \Gamma(X, E) \longrightarrow \Gamma(x_0, E_{x_0}) \cong B$$

is a fibration.

Proof. Take $X' = \{x_0\}$. \square

⁵i.e., the inclusion $x_0 \rightarrow X$ is a cofibration.

Proposition 6.3. *Suppose that $E \rightarrow X$ is a standard bundle of Banach algebras over a compact space X . Let X' be a closed subspace and assume that the inclusion $X' \rightarrow X$ is a cofibration. Let $E' \rightarrow X'$ be the induced bundle. Define $\mathrm{GL}_\circ E$ to be the union of the various $\mathrm{GL}_\circ(E_x)$ topologized as a subspace of E . Then*

- (1) $\mathrm{GL}_\circ E \rightarrow X$ is a fibration.
- (2) The induced map

$$r : \Gamma(X, \mathrm{GL}_\circ E) \rightarrow \Gamma(X', \mathrm{GL}_\circ E')$$

is a fibration.

Proof. This is a simple pullback argument using the fact that when moving from E to E' the map is an isomorphism on fibres. \square

Remark 6.4. We use $F_\bullet(X, Y) \subset F(X, Y)$ to denote basepoint-preserving maps, $\Gamma_\bullet(X, E) \subset \Gamma(X, E)$ to denote basepoint-preserving sections, and similarly for subgroups of the various general linear groups. The identity element of a group is always taken to be its basepoint.

Corollary 6.5. *Suppose that $E \rightarrow X$ is a locally trivial fibration of Banach algebras over a compact space X . Let $x_0 \in X$ be a non-degenerate basepoint. Then the induced evaluation map*

$$r : \Gamma(X, \mathrm{GL}_\circ E) \rightarrow \Gamma(x_0, E_{x_0}) \cong \mathrm{GL}_\circ B$$

is a fibration. Further, r is a homomorphism of topological groups and

$$\mathrm{Ker}(r) = \{\sigma \in \Gamma(X, \mathrm{GL}_\circ E) : \sigma(x_0) = 1\} \equiv \Gamma_\bullet(X, \mathrm{GL}_\circ E).$$

We now return to our main focus. We summarize:

Theorem 6.6. *Let $\zeta : P \rightarrow X$ be a principal G bundle over a compact space X and let G act on a Banach algebra B . Form the associated fibre bundle $E = P \times_G B \rightarrow X$ and let $A_\zeta \equiv \Gamma(X, E)$. Then the evaluation homomorphism $r : \mathrm{GL}_\circ A_\zeta \rightarrow \mathrm{GL}_\circ B$ yields a fibration*

$$\mathrm{GL}_\circ \bullet A_\zeta \rightarrow \mathrm{GL}_\circ A_\zeta \xrightarrow{r} \mathrm{GL}_\circ B$$

with fibre $\mathrm{GL}_\circ \bullet A_\zeta$.

This result is closely related to the C^* -algebraic results of K. Thomsen ([24], cf. Theorem 1.9.)

7. ENHANCED SAMELSON PRODUCTS

Let G be a topological group and let Y_1 and Y_2 be topological spaces. The traditional Samelson product (cf. [19], [27] p. 467, [16] §6.3) is a pairing

$$[\ , \] : [Y_1, G] \times [Y_2, G] \rightarrow [Y_1 \wedge Y_2, G]$$

defined by

$$[[\phi], [\psi]] = [\eta] \quad \text{where} \quad \eta(w \wedge y) = \phi(w)\psi(y)\phi(w)^{-1}\psi(y)^{-1}.$$

If $Y_1 = S^r$ and $Y_2 = S^s$ this gives a pairing

$$[\ , \] : \pi_r(G) \times \pi_s(G) \rightarrow \pi_{r+s}(G).$$

We wish to generalize this construction to our context.

Definition 7.1. Suppose that B is a Banach algebra with a continuous group action given by a map $\alpha : G \rightarrow \text{Aut}(B)$. We define the *enhanced Samelson product*

$$[Y_1, G] \times [Y_2, \text{GL } B] \longrightarrow [Y_1 \wedge Y_2, \text{GL } B]$$

by

$$[[\phi], [\psi]] = [\eta] \quad \text{where} \quad \eta(w \wedge y) = \alpha_{\phi(w)}(\psi(y))\psi(y)^{-1}.$$

Taking $Y_1 = S^r$ and $Y_2 = S^s$ gives an *enhanced Samelson product*

$$[\ , \] : \pi_r(G) \times \pi_s(\text{GL } B) \longrightarrow \pi_{r+s}(\text{GL } B).$$

Remark 7.2. If the action of G on B is inner then we write the action of G on B as

$$(g, b) \longrightarrow gbg^{-1}.$$

Then

$$\alpha_{\phi(w)}(\psi(y)) = \phi(w)\psi(y)\phi(w)^{-1}$$

and hence

$$[[\phi], [\psi]] = [\eta] \quad \text{where} \quad \eta(w \wedge y) = \phi(w)\psi(y)\phi(w)^{-1}\psi(y)^{-1}$$

which is the traditional Samelson formula. So the enhanced Samelson product is a true generalization of the classical Samelson product..

Remark 7.3. If B is a C^* -algebra and G is locally compact then (B, G) form what G. Pedersen [17] calls a C^* -dynamical system. Pedersen shows ([17], p. 257) that there is a Hilbert space \mathcal{H} upon which G acts by the regular representation λ and a faithful covariant representation ρ of (B, G) of the dynamical system. Thus up to isomorphism we may replace (B, G) by $(\rho(B), G)$. Then

$$\rho(\alpha_g(b)) = \lambda_g \rho(b) \lambda_g^{-1}.$$

So in this case too the enhanced Samelson product reduces down to a commutator of the form $U_g b U_g^{-1} b^{-1}$ as well, even though U_g is not in $\text{GL } B$.

8. IDENTIFYING THE DIFFERENTIAL MORE PRECISELY

We introduce some notation in order to analyze the situation over spheres. Let $\zeta : P \rightarrow S^k$ denote a principal G -bundle. It is classified by its clutching map $\kappa : S^{k-1} \rightarrow G$ which we realize explicitly as follows.

Let D^n denote the n -ball, regarded as the cone on S^{n-1} :

$$D^n = [0, 1] \times S^{n-1} / \{0\} \times S^{n-1}$$

and we write $(t, x) \rightarrow tx$. We decompose the base space S^k as the disjoint union $S^k = H^+ \cup H^-$ of upper and lower closed hemispheres, with equator $S^{k-1} = H^+ \cap H^-$. The restriction of the principal bundle $\zeta : P \rightarrow S^k$ to each closed hemisphere trivializes, and so there are sections

$$\sigma^\pm : H^\pm \rightarrow H^\pm \times G \quad \sigma^\pm(x) = (x, s^\pm(x))$$

and a clutching map $\kappa : S^{k-1} \rightarrow G$ satisfying

$$s^+(x) = \kappa(x) s^-(x) \kappa(x)^{-1} \quad \forall x \in S^{k-1}$$

that determine the principal bundle ζ up to equivalence. The triviality of ζ over each hemisphere implies that the associated bundle $P \times_G \mathrm{GL}_o B \rightarrow S^k$ is also trivial over each hemisphere. Thus we may alternately describe $\mathrm{GL}_o(A_\zeta)$ as

$$\mathrm{GL}_o' A_\zeta = \{(s^+, s^-) : H^+ \sqcup H^- \rightarrow G : s^+(x) = \kappa(x)s^-(x)\kappa(x)^{-1} \quad \forall x \in S^{k-1}\}$$

where we identify $\mathrm{GL}_o(A_\zeta) \cong \mathrm{GL}_o'(A_\zeta)$ by

$$s \rightarrow (s\sigma^+, s\sigma^-).$$

Taking the south pole x_o as basepoint of S^k , the evaluation map

$$r : \Gamma(P \times_G \mathrm{GL}_o B \rightarrow S^k) \longrightarrow \mathrm{GL}_o B$$

is given in this picture by $r(s^+, s^-) = s^-(x_o)$.

The following proposition would seem to be folklore.

Proposition 8.1. *There is a natural identification*

$$\pi_n(\mathrm{GL}_o \bullet A_\zeta) \cong \pi_n(F_\bullet(S^k, \mathrm{GL}_o B)) \cong \pi_{n+k}(\mathrm{GL}_o B).$$

Proof. Let

$$\mathrm{GL}_o'' A_\zeta = \{(s^+, s^-) \in \mathrm{GL}_o' A_\zeta : s^-(x) = e \quad \forall x \in H^-\}.$$

It is an exercise (cf. Wockel [28] Lemma 4.1.6) to show that the natural inclusion

$$\mathrm{GL}_o'' A_\zeta \longrightarrow \mathrm{GL}_o' A_\zeta$$

is a homotopy equivalence. But then it is easy to see that

$$\mathrm{GL}_o'' A_\zeta = \{s^+ : H^+ \rightarrow \mathrm{GL}_o B : s^+(x) = e \quad \forall x \in \partial(H^+)\} \cong F_\bullet(S^k, \mathrm{GL}_o B)$$

so the proposition is immediate. \square

The following result generalizes Wockel [29], Theorem 2.3 and we have adapted his proof as well.

Theorem 8.2. *Let $\zeta : P \rightarrow S^k$ be a principal G -bundle with clutching map $\kappa : S^{k-1} \rightarrow G$. Let $A_\zeta = \Gamma(P \times_G B \rightarrow S^k)$ denote the associated Banach algebra, with associated evaluation fibration*

$$\mathrm{GL}_o \bullet A_\zeta \longrightarrow \mathrm{GL}_o A_\zeta \xrightarrow{r} \mathrm{GL}_o B.$$

Let ∂ denote the boundary homomorphism in the long exact homotopy sequence associated to the evaluation fibration and let δ_n denote the composition

$$\pi_n(\mathrm{GL}_o B) \xrightarrow{\partial} \pi_{n-1}(\mathrm{GL}_o \bullet A_\zeta) \cong \pi_{n-1}(F_\bullet(S^k, \mathrm{GL}_o B)) \cong \pi_{n+k-1}(\mathrm{GL}_o B).$$

Then δ_n is given by

$$\delta_n(a) = -\alpha_\kappa a^{-1} \equiv -[\kappa, a]$$

where $[\kappa, a]$ is the enhanced Samelson product.

Proof. Our proof follows Wockel [29], Theorem 2.3 in spirit. Represent a by $a : [0, 1] \times S^{n-1} \rightarrow \mathrm{GL}_o B$ with a trivial on $\{0, 1\} \times S^{n-1}$. Without loss of generality we may also assume that it is trivial on $[0, 1] \times \{x_o\}$. Define maps A^\pm as follows:

$$\begin{aligned} A^+ : D^n \times H^+ &\rightarrow \mathrm{GL}_o B & A^+(d, tx) &= \alpha_{\kappa(x)}(a(t(d))) \\ A^- : D^n \times H^- &\rightarrow \mathrm{GL}_o B & A^-(d, y) &= a(d). \end{aligned}$$

If $t = 1$ then $A^+(d, x) = \alpha_{\kappa(x)}(a(d))$ and $A^-(d, y) = a(d)$ as desired, and so the maps patch together to form (after taking adjoints) a map

$$A : D^n \longrightarrow \mathrm{GL}_o A_\zeta$$

with the property that

$$ev(A(d)) = A^-(d, 0) = a(d).$$

Now collapse all of H^- to the south pole, the basepoint of S^k . The result is another copy of S^k , of course and by definition δ_n is given by

$$\delta_n(a) = [A^+|_{\partial D^n \times D^k}] \in [\partial D^n \times S^k, \mathrm{GL}_o B]_*$$

Now define $\tilde{A} : D^n \times D^k \rightarrow \mathrm{GL}_o B$ by

$$\tilde{A}(d, x) = A^+(d, x)a(d)^{-1}$$

Then:

- (1) $\tilde{A} = *$ on $\partial D^n \times \partial D^k$.
- (2) $\tilde{A} = *$ on $D^n \vee D^k$.
- (3) $\tilde{A} = A$ on $\partial D^n \times D^k$ because a is trivial there.
- (4) $\tilde{A}(d, x) = [\kappa, a]$ on $D^n \times \partial D^k$ since $t = 1$ there.

Thus

$$\delta_n(a) = [A^+|_{\partial D^n \times D^k}] = [\tilde{A}|_{\partial D^n \times D^k}] = -[\tilde{A}|_{D^n \times \partial D^k}]$$

$$\text{since } [\tilde{A}|_{\partial(D^n \times D^k)}] = 0$$

$$= -[\kappa, a].$$

To complete the proof we note that the entire spectral sequence is natural under localization. □

9. PROOFS OF THE MAIN THEOREMS

We have already done the heavy lifting for Theorem A. Here is the conclusion:

Proof of Theorem A.

Proof. It suffices to observe that the diagram

$$\begin{array}{ccc} \pi_n(\mathrm{GL}_o B) \otimes \mathbb{P} & \xrightarrow{\delta_k} & \pi_{n+k-1}(\mathrm{GL}_o B) \otimes \mathbb{P} \\ \cong \downarrow & & \cong \downarrow \\ H^0(S^k) \otimes \pi_n(\mathrm{GL}_o B) \otimes \mathbb{P} & \xrightarrow{d_k} & H^k(S^k) \otimes \pi_{n+k-1}(\mathrm{GL}_o B) \otimes \mathbb{P} \end{array}$$

commutes, and this is evident. □

Theorem B is mostly immediate, except for the special case $k = 3$ and we turn our attention to that case now.

Specialize to the case $k = 3$ and $B = \mathcal{K}$, the compact operators on the standard Hilbert space \mathcal{H} . In this case the (contractible) unitary group $\mathcal{U} = \mathcal{U}(\mathcal{H})$ acts on \mathcal{K} by conjugation, its center S^1 acts trivially, of course, and hence the action descends to an action of the projective unitary group \mathcal{PU} on \mathcal{K} . Note that $\mathcal{PU} \simeq BS^1 \simeq K(\mathbb{Z}, 2)$.

Let

$$\mathcal{UK} = \{u \in \mathcal{U}(\mathcal{H}) : u - 1 \in \mathcal{K}\},$$

the unitary group of \mathcal{K} . Recall that $\Omega^2 \mathcal{UK} \simeq \mathcal{UK}$ is one way of stating Bott periodicity.

Suppose that $\zeta : P \rightarrow S^3$ is a principal \mathcal{PU} -bundle. This bundle is classified by a map $\kappa : S^2 \rightarrow \mathcal{PU}$ as per the notation above, and the homotopy class of κ in $[S^2, \mathcal{PU}] \cong H^2(S^2; \mathbb{Z})$ has the form $\Delta_\zeta g$ (where g is a generic term for canonical generator) and so determines an integer Δ_ζ which is essentially the Dixmier-Douady class of the principal bundle.

Let $A_\zeta = \Gamma(P \times_{\mathcal{PU}} \mathcal{K} \rightarrow S^3)$ denote the associated Banach algebra. Then a consequence of the Theorem is that $\pi_*(UA_\zeta)$ is determined by

$$\delta_3 : \pi_n(\mathcal{UK}) \longrightarrow \pi_{n+2}(\mathcal{UK})$$

These groups are zero for n even and \mathbb{Z} for n odd by Bott periodicity. If we see this in terms of E_2 it corresponds to

$$d_3 : H^0(S^3) \otimes \pi_n(\mathcal{UK}) \longrightarrow H^3(S^3) \otimes \pi_{n+2}(\mathcal{UK})$$

and we have proved that

$$d_3(1 \otimes a) = -g[\kappa, a].$$

Proposition 9.1. *With the notation above, for n odd,*

$$[\kappa, a] = \Delta_\zeta \hat{\beta}(a)$$

where $\hat{\beta}(a)$ is the composite

$$S^{n+2} \simeq S^2 \wedge S^n \xrightarrow{1 \wedge a} S^2 \wedge \mathcal{UK} \xrightarrow{\hat{\beta}} \mathcal{UK}$$

and $\hat{\beta} : S^2 \wedge \mathcal{UK} \rightarrow \mathcal{UK}$ is the adjoint of the Bott periodicity identification $\beta : \mathcal{UK} \simeq \Omega^2 \mathcal{UK}$.

Proof. Regard $\kappa : S^2 \rightarrow \mathcal{PU} \simeq BS^1$ as a line bundle $L \rightarrow S^2$ with first Chern class $c_1(L) = \Delta_\zeta g$. Regard $a : S^n \rightarrow \mathcal{UK}$ as the clutching map of a \mathcal{K} -bundle $F \rightarrow S^{n+1}$. Then the bundle $L \otimes F \rightarrow S^2 \wedge S^{n+1} \cong S^{n+3}$ is represented by the clutching map

$$\kappa(x)a(y)\kappa(x)^{-1} : S^2 \wedge S^n \longrightarrow \mathcal{UK}$$

Then we appeal to the argument of Proposition 2.1 of Atiyah-Segal [3]. They note that $L \otimes F$ is a sub-bundle of the trivial bundle $\mathcal{H} \otimes F$ and hence

$$\kappa(x)a(y)\kappa(x)^{-1}a(y)^{-1} = \Delta_\zeta(1 \wedge a(y)^{-1}).$$

Using notation introduced previously, we rewrite this as

$$[\kappa, a] = \Delta_\zeta \hat{\beta}(a)$$

and this proves the proposition. □

Proof of Theorem B.

Proof. This is mostly immediate by taking direct limits from A. The only remaining issue is the explicit identification of the differential in the case $k = 3$ and that is done in the previous proposition. □

10. AN EXAMPLE

In this section we illustrate our result in a very concrete case. Take the principal bundle to be the Hopf bundle

$$\zeta : S^7 \longrightarrow S^4$$

obtained from the multiplicative structure of the quaternions, with group $G = S^3 = SU_2$. Take $B = M_2(\mathbb{C})$ with S^3 acting upon $M_2(\mathbb{C})$ by conjugation. Then there is an associated bundle of C^* -algebras

$$S^7 \times_{S^3} M_2(\mathbb{C}) \longrightarrow S^4$$

and as usual we denote by A_ζ the associated C^* -algebra of continuous sections. We want to compute $\pi_*(UA_\zeta)$. The Wang sequence then takes the form

$$\longrightarrow \pi_n(UA_\zeta) \longrightarrow \pi_n(U_2) \xrightarrow{d_4} \pi_{n+3}(U_2) \longrightarrow \pi_{n-1}(UA_\zeta) \longrightarrow$$

Recall that $U_2 \cong S^1 \times S^3$ as topological spaces, though not as groups. Serre's classical results on homotopy imply that $\pi_n(U_2)$ is a finite group for each $n > 3$ and that these groups are non-zero for infinitely many values of n .

We record for reference the first twelve homotopy groups of U_2 . Let $\eta \in \pi_3(S^2)$ denote the Hopf generator and by abuse of notation its various suspensions, so for instance we write $\eta^2 \in \pi_5(S^3)$ for the composition

$$S^5 \xrightarrow{S^2\eta} S^4 \xrightarrow{S\eta} S^3.$$

We use a_n as labels for classes when there don't seem to be standard names; the subscript denotes the homotopy group.

- $\pi_1(U_2) \cong \pi_1(S^1) \cong \mathbb{Z}$ on the class of the upper left corner inclusion $S^1 \rightarrow U_2$ which we denote a_1 . In all higher degrees the natural map $S^3 \cong SU_2 \rightarrow U_2$ induces an isomorphism in homotopy.
- $\pi_2(U_2) = 0$.
- $\pi_3(U_2) \cong \mathbb{Z}$. The generator is given by the canonical inclusion (slight abuse of notation)

$$\iota : S^3 \cong SU_2 \rightarrow U_2.$$

- $\pi_4(U_2) \cong \mathbb{Z}/2$ on the class η .
- $\pi_5(U_2) \cong \mathbb{Z}/2$ on the class η^2 .
- $\pi_6(U_2) \cong \mathbb{Z}/12$ on the class a_6 . The 2-primary part is generated by a class ν' (in Toda's [25] notation).
- $\pi_7(U_2) \cong \mathbb{Z}/2$ on the class $\nu'\eta$.
- $\pi_8(U_2) \cong \mathbb{Z}/2$ on the class $\nu'\eta^2$.
- $\pi_9(U_2) \cong \mathbb{Z}/3$ on the class a_9 . (J. C. Moore [15], Theorem 5.3.)
- $\pi_{10}(U_2) \cong \mathbb{Z}/15$ on the class a_{10} . (J. C. Moore [15], Theorem 5.3 and Lemma 5.1.)
- $\pi_{11}(U_2) \cong \mathbb{Z}/2$ (Toda [25] Theorem 7.2).

- $\pi_{12}(U_2) \cong (\mathbb{Z}/2)^2$ (Toda [25] Theorem 7.2).

Recall that we have shown that the differential d_4 is given by

$$d_4(a) = -g \otimes [\kappa, a]$$

where κ is the clutching map of the principal bundle. In this example, we have:

Proposition 10.1. *The clutching map of the Hopf bundle $S^7 \rightarrow S^4$ is the identity map $\iota : S^3 \rightarrow S^3$.*

We are indebted to John Klein for the following proof of this fact.

Proof. This fact is a general characteristic of the Hopf construction, in the case $G = S^3$ with its standard multiplication. If G is a topological group with multiplication $G \times G \rightarrow G$, one has a Hopf construction

$$G \star G \rightarrow SG$$

where \star denotes join and SG is the unreduced suspension of G . This is a fibre bundle with fibre at the basepoint G . The bundle projection is given by

$$tg + (1-t)h \rightarrow t(gh) \quad t \in [0, 1], \quad g, h \in G.$$

The clutching map in this case is given by the map $G \rightarrow \text{homeo}(G)$ which is adjoint to left multiplication. This factors through the identity map of G considered as acting by left multiplication on itself which shows that the fibration has a reduction of structure group to G and has clutching map $\iota : G \rightarrow G$. \square

In light of the Proposition, we see that in our example the differential is given by

$$d_4(a) = -g \otimes [\iota, a]$$

where $\iota : S^3 \rightarrow S^3$ is the identity map. So we must calculate the Samelson product

$$[\iota, -] : \pi_n(U_2) \rightarrow \pi_{n+3}(U_2).$$

Here is the result. Note that each entry that is non-zero corresponds to a non-zero d_4 differential.

- $n = 1$: $[\iota, a_1] = 0$
- $n = 3$: $[\iota, \iota] = a_6$ by the result of I. M. James [10], p. 176.
- $n = 4$: $[\iota, \eta] = \nu'\eta$ since (working 2-primary)
$$[\iota, \eta] = [\iota, \iota] \circ \eta = \nu'\eta.$$
- $n = 5$: $[\iota, \eta^2] = \nu'\eta^2$ by the same argument.
- $n = 6$: $[\iota, a_6] = [\iota, [\iota, \iota]] = a_9$ by I. M. James [9], §3.
- $n = 7$: $[\iota, \nu'\eta] = [\iota, \nu'] \circ \eta = a_9 \circ \eta = 0$ (since a_9 has order 3 and η has order 2.)
- $n = 8$: $[\iota, \nu'\eta^2] = 0$ by same argument.
- $n = 9$: $[\iota, a_9] = 0$ by nilpotency.

Feeding this information into the Wang long exact sequence produces the following results:

Theorem 10.2. *Let*

$$\zeta : S^7 \longrightarrow S^4$$

denote the Hopf bundle. Form the associated bundle of C^ -algebras*

$$S^7 \times_{S^3} M_2(\mathbb{C}) \longrightarrow S^4$$

and let A_ζ denote the associated C^ -algebra of continuous sections. Then $\pi_*(U_\bullet A_\zeta)$ is given in low dimensions as follows:*

- $\pi_1(U_o A_\zeta)$ fits into a split short exact sequence

$$0 \rightarrow \pi_5(U_2) \longrightarrow \pi_1(U_o A_\zeta) \longrightarrow \pi_1(U_2) \rightarrow 0$$

with $\pi_5(U_2) \cong \mathbb{Z}/2$ and $\pi_1(U_2) \cong \mathbb{Z}$ and so $\pi_1(U_o A_\zeta) = \mathbb{Z} \oplus \mathbb{Z}/2$.

- $\pi_2(U_o A_\zeta) = 0$.

- $\pi_3(U_o A_\zeta)$ fits in a short exact sequence

$$0 \rightarrow \pi_3(U_o A_\zeta) \longrightarrow \pi_3(U_2) \longrightarrow \pi_6(U_2) \rightarrow 0$$

with $\pi_3(U_2) \cong \mathbb{Z}$ and $\pi_6(U_2) \cong \mathbb{Z}/12$, and so $\pi_3(U_o A_\zeta) \cong \mathbb{Z}$.

- $\pi_4(U_o A_\zeta) = 0$.

- $\pi_5(U_o A_\zeta) = 0$.

- $\pi_6(U_o A_\zeta)$ fits in a short exact sequence

$$0 \rightarrow \pi_{10}(U_2) \longrightarrow \pi_6(U_o A_\zeta) \longrightarrow \mathbb{Z}/4 \rightarrow 0$$

with $\pi_{10}(U_2) \cong \mathbb{Z}/15$ and $\mathbb{Z}/4$ the 2-primary component of $\pi_6(U_2)$, so that $\pi_6(U_o A_\zeta) \cong \mathbb{Z}/60$.

- $\pi_7(U_o A_\zeta)$ fits in a short exact sequence

$$0 \rightarrow \pi_{11}(U_2) \longrightarrow \pi_7(U_o A_\zeta) \longrightarrow \pi_7(U_2) \rightarrow 0$$

with $\pi_{11}(U_2) \cong \mathbb{Z}/2$ and $\pi_7(U_2) \cong \mathbb{Z}/2$. So $\pi_7(U_o A_\zeta)$ is a group with four elements (and there may be a non-trivial group extension).

- $\pi_8(U_o A_\zeta)$ fits in a short exact sequence

$$0 \rightarrow \pi_{12}(U_2) \longrightarrow \pi_8(U_o A_\zeta) \longrightarrow \pi_8(U_2) \rightarrow 0$$

with $\pi_{12}(U_2) \cong (\mathbb{Z}/2)^2$ and $\pi_8(U_2) \cong \mathbb{Z}/2$. So $\pi_8(U_o A_\zeta)$ is a group with eight elements (and there may be a non-trivial group extension).

We contrast this with the analogous computation in rational homotopy and in K -theory.

Theorem 10.3. *With the notation above,*

- (1) *The rational homotopy groups of $U_o A_\zeta$ are zero except for*

$$\pi_j(U_o A_\zeta) \otimes \mathbb{Q} \cong \mathbb{Q} \quad j = 1 \text{ and } 3.$$

- (2) *The (matrix) stable homotopy groups are zero in even degrees and*

$$\pi_j(U_o(A_\zeta \otimes \mathcal{K})) \cong \mathbb{Z} \quad j \text{ odd}$$

- (3) *The K -theory groups are given by $K_0(A_\zeta) \cong \mathbb{Z}$ and $K_1(A_\zeta) = 0$.*

Proof. In the rational homotopy case the only non-zero homotopy groups of U_2 are

$$\pi_1(U_2) \otimes \mathbb{Q} \cong \pi_3(U_2) \otimes \mathbb{Q} \cong \mathbb{Q}.$$

Since the Wang differential changes degree by three, it must be identically zero, the spectral sequence collapses, and the long exact Wang sequence turns into many short exact sequences. As $\pi_j(U_2) = 0$ except for $j = 1, 3$, the result is as stated.

In the stable case the situation is similar, since stably $\pi_*(U_o(M_2 \otimes \mathcal{K})) \cong \mathbb{Z}$ for $*$ odd and zero for $*$ even by Bott periodicity. Again, the Wang differential is identically zero and the result follows. Part (3) follows from (2) essentially by definition. \square

We regard this example as an excellent illustration of what is lost by focusing attention only upon $K_*(A_\zeta)$. The richness of detail that is evident while studying the individual homotopy groups is completely lost upon matrix stabilization and passage to K -theory.

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